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A NOTE ON A PROBLEM OF ERDÖS AND HAJNAL

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In [5], Erdős and Hajnal formulate the following proposition, which we shall refer to as Φ: If φ is an order-type such that $|\varphi| = \omega_2$ but $\omega_2, \omega_2^* \not\leqslant \varphi$, there is $\psi \leqslant \varphi$, $|\psi| = \omega_1$, such that $\omega_1, \omega_1^* \not\leqslant \psi$. In [2], we showed that if $V = L$, then $\neg\Phi$. We do not know if the assumption $V = L$ can be weakened to CH, or if, in fact, Φ is consistent with CH. However, in this note we show that, relative to a certain large cardinal assumption, Φ is consistent with $2^\omega = \omega_2$, so that $\neg\Phi$ is not provable in ZFC alone. Our proof has an interesting model-theoretic consequence, which we mention at the end.

1. Preliminaries

We work in ZFC, and use the usual notation and conventions. In particular, an ordinal is the set of its predecessors, a cardinal is an ordinal not equinumerous with any smaller ordinal, α, β, γ denote ordinals, κ, λ, μ denote cardinals, and $|X|$ denotes the cardinality of the set X. We assume considerable acquaintance with forcing, as described in [6] for example, and also some familiarity with indiscernibility arguments using large cardinals.

A set $X \subseteq \kappa$ is said to be *homogeneous* for the first-order structure $\mathfrak{A} = \langle A, ... \rangle$, where $\kappa \subseteq A$, if for all formulas $\varphi(v_0, ..., v_n)$ in the language for $\mathfrak{A}$, if $x_0, ..., x_n, x_0', ..., x_n' \in X$, $x_0 < ... < x_n$, $x_0' < ... < x_n'$, then $\mathfrak{A} \models \varphi[x_0, ..., x_n]$ iff $\mathfrak{A} \models \varphi[x_0', ..., x_n']$.

A cardinal κ is *Ramsey* iff whenever $\mathfrak{A} = \langle A, ... \rangle$ is a first-order structure such that $\kappa \subseteq A$ and the language of $\mathfrak{A}$ has less than κ symbols, there is $X \subseteq \kappa$, $|X| = \kappa$, X homogeneous for $\mathfrak{A}$. For further details, the reader should consult Drake [4].

A cardinal κ is *weakly compact* iff whenever $\varphi(U, W_1, ..., W_n)$ is a sentence in the language of set theory augmented by the unary predicate letters $U, W_1, ..., W_n$, such that for some $W_1, ..., W_n \subseteq V_\kappa$, $\langle V_\kappa, \in, U, W_1, ..., W_n \rangle \models \varphi$ for all $U \subseteq V_\kappa$, then for some $\alpha < \kappa$,

$\langle V_\alpha, \in, U, W_1 \cap V_\alpha, ..., W_n \cap V_\alpha \rangle \models \varphi$ for all $U \subseteq V_\alpha$. Again, [4] will provide further details here. For our present purposes we need to know that every Ramsey cardinal is weakly compact, and that every weakly compact is a fixed-point in the sequence of all inaccessible cardinals. (We assume the reader is well aware of what an inaccessible cardinal is, and also what a weakly inaccessible cardinal is. If he doesn't, he would be much better off reading [4] than the present paper.) *Chang's conjecture*, which we shall denote by Δ, is the assertion that if we are given a first-order structure $\mathfrak{A} = \langle A, U, ... \rangle$, where $|A| = \omega_2$, $U \subseteq A$, $|U| = \omega_1$, and the language for $\mathfrak{A}$ is countable, we can find $\mathfrak{B} = \langle B, U \cap B, ... \rangle \prec \mathfrak{A}$ such that $|B| = \omega_1$, $|U \cap B| = \omega$. It is known that Δ is not provable in ZFC. In fact, it follows easily from the results proved towards the end of chapter 17 of [1] that Δ implies the existence of $0^\#$ (which is defined in [1, chapter 17]). This was first proved by Kunen. Also, Silver [8] has shown that

$$\mathrm{Con}(\mathrm{ZFC} + \text{"there is a Ramsey cardinal"}) \rightarrow \mathrm{Con}(\mathrm{ZFC} + \Delta).$$

2. Basic forcing lemmas

We use M to denote throughout an arbitrary countable transitive model (c.t.m.) of ZFC. For proofs of all of the following lemmas, the reader should consult [6].

Lemma 2.1 (Lévy, Solovay, et al.). *Let κ be inaccessible/weakly compact/Ramsey in M. Let P be a poset in M of cradinality less than κ. If G is M-generic for P, then κ is inaccessible/weakly compact/Ramsey in $M[G]$.*

Lemma 2.2 (Solovay). *Let P_1, P_2 be posets in M. If G_1 is M-generic for P_1 and G_2 is $M[G_1]$-generic for P_2, then G_1 is $M[G_2]$-generic for P_1, G_2 is M-generic for P_2, $G_1 \times G_2$ is M-generic for $P_1 \times P_2$, and $M[G_1][G_2] = M[G_2][G_1] = M[G_1, G_2] = M[G_1 \times G_2]$, where $P_1 \times P_2$ is the cartesian product of the sets P_1, P_2 with the ordering $\langle p_1, p_2 \rangle \leqslant \langle q_1, q_2 \rangle \leftrightarrow p_1 \leqslant_1 q_1 \;\&\; p_2 \leqslant_2 q_2$. Conversely, if G is M-generic for $P_1 \times P_2$, then $G_1 = \{p \mid \langle p, 1 \rangle \in G\}$ is M-generic for P_1, $G_2 = \{p \mid \langle 1, p \rangle \in G\}$ is $M[G_1]$-generic for P_2, and $G = G_1 \times G_2$. (As usual, we assume our posets have a maximum element, 1.)*

Lemma 2.3 (Solovay). *Let P_1, P_2 be sets in M. Let $\leqslant_1$ be a partial ordering of P_1 in M and let $\leqslant_2$ be a term of the $(M, \langle P_1, \leqslant_1 \rangle)$-forcing language such that $1 \Vdash_{P_1}$ "$\leqslant_2$ is a partial ordering of $\check{P}_2$". Define, in M, a partial ordering on $P_1 \times P_2$ by $\langle p_1, p_2 \rangle \leqslant \langle q_1, q_2 \rangle \leftrightarrow p_1 \leqslant_1 q_1$ & $p_1 \Vdash_{P_1}$ "$\check{p}_2 \leqslant_2 \check{q}_2$". If G_1 is M-generic for P_1 and G_2 is $M[G_1]$-generic for P_2 (i.e., the poset $\langle P_2, \leqslant_2^{M[G_1]} \rangle$ in $M[G_1]$), then $G_1 \times G_2$ is M-generic for $P_1 \times P_2$. Conversely, if G is M-generic for $P_1 \times P_2$, there are sets G_1, G_2 such that G_1 is M-generic for P_1, G_2 is $M[G_1]$-generic for P_2, and $G = G_1 \times G_2$.*

Recall that a poset P has the κ *chain condition* (κ-c.c.) if there is no pairwise incompatible subset of P of cardinality κ, and that ω_1-c.c. is referred to as the *countable chain condition* (c.c.c.). (We say $p, q \in P$ are *compatible* if there is $r \in P$, $r \leqslant p, q$, and write $p \sim q$ in such a situation.)

Lemma 2.4. *Let P be a poset satisfying c.c.c. in M, and let G be M-generic for P. Then M and $M[G]$ have the same cardinals and cofinality function.*

Martin's Axiom for ω_1 is the assertion that if P is a poset with c.c.c. and $\mathcal{D}$ is a collection of ω_1 dense open subsets of P, there is a $\mathcal{D}$-generic set G for P. We denote this statement by MA. It is easily seen that MA $\to 2^\omega \geqslant \omega_2$.

Lemma 2.5 (Solovay and Jennenbaum). *Suppose $M \models 2^\omega = \omega_1$. Then there is a poset $P \in M$ of cardinality ω_2, satisfying c.c.c., such that for any set G, M-generic for P, $M[G] \models$* MA + $2^\omega = \omega_2$.

That completes our list of prerequisites. It is convenient at this point to set out our plan of attack.

3. The strategy

In [2], we prove the following theorem:

Theorem 3.1 (Devlin). *Assume Δ. If $\neg\Phi$, then there is an ω_2-Aronszajn tree.*

It thus suffices, for our purposes, to show that

Con(ZFC + "there is a Ramsey cardinal") →

Con(ZFC + $2^\omega = \omega_2$ + Δ + "there are no ω_2-Aronszajn trees").

Now, in [8], Silver proves

Con(ZFC + "there is a Ramsey cardinal") → Con(ZFC + $2^\omega = \omega_1$ + Δ).

Since $2^\omega = \omega_1$ in Silver's model, it contains an ω_2-Aronszajn tree (which remains an ω_2-Aronszajn tree in any cardinal preserving extension of it.)

Hence Silver's model does not help us here. Again, in [7], Mitchell [1] proves

Con(ZFC + "there is a weakly compact cardinal") →

Con(ZFC + $2^\omega = \omega_2$ + "there are no ω_2-Aronszajn trees").

The idea behind our proof is to combine the proofs of Mitchell and of Silver. In order to do this, we have to make some considerable changes in both proofs, so, even though the overall plan remains a combination of the Mitchell argument and the Silver argument, we see no alternative but to give most of the proof in full. In several places, the argument will be exactly parallel to Mitchell's (in particular), and at such points we shall leave it to the reader to check that Mitchell's argument indeed works in the present situation. This will not require that the reader is familiar with all of Mitchell's paper; indeed, he should be able to simply read the proof concerned and see that, with a few minor changes, it does what we require.

For readers who are familiar with [7], let us state now that the difference between our model and Mitchell's lies in the way the continuum is collapsed to ω_2.

4. The proof

From now on, we fix κ as the first Ramsey cardinal in M. Define C in M as the poset of all finite functions p such that $\mathrm{dom}(p) \subseteq \kappa$ and $\mathrm{ran}(p) \subseteq 2$, ordered by $p \leqslant_c q \leftrightarrow p \supseteq q$. Thus, C is the usual poset for

[1] Strictly speaking, this result is due jointly to Mitchell and Silver. However, most of the proof is due to Mitchell. What Silver actually proved was the analogue of our Theorem 4.10.

adding κ Cohen reals to M. If G is M-generic for P (which it will be from now on), then $2^{\omega} = \kappa$ in $M[G]$. Also, as C satisfies c.c.c. in M, M and $M[G]$ have the same cardinals and cofinality function. in particular, κ is weakly inaccessible and is the limit of a κ-sequence of weakly inaccessibles. In fact, if $\langle\kappa(\nu) \mid \nu < \kappa\rangle$ enumerates (monotonically) the weakly inaccessible cardinals below κ *in* M, then each $\kappa(\nu)$ is weakly inaccessible in $M[G]$. Note also that the definition of C is absolute for transitive models of ZFC containing κ. For $\gamma < \kappa$, we set $C_\gamma = \{p \in C \mid \mathrm{dom}(p) \subseteq \gamma\}$, $C^\gamma = \{p \in C \mid \mathrm{dom}(p) \cap \gamma = \emptyset\}$. Since we clearly have $C \cong C_\gamma \times C^\gamma$, by a canonical isomorphism (in M), we see that $G_\gamma = G \cap C_\gamma$ is M-generic for C_γ, $G^\gamma = G \cap C^\gamma$ is $M[G_\gamma]$-generic for C^γ, $M[G_\gamma][G^\gamma] = M[G]$, and all of the other properties in Lemma 2.2 hold.

Let $\mathbf{B}$ be the complete boolean algebra determined by C, isomorphed so that C is a dense subset of $\mathbf{B}$. For each $\gamma < \kappa$, let $\mathbf{B}_\gamma$ be the complete boolean algebra determined by C_γ, isomorphed so that $\gamma < \delta < \kappa$ implies that $\mathbf{B}_\gamma$ is a complete subalgebra of $\mathbf{B}_\delta$ is a complete subalgebra of $\mathbf{B}$.

In M, let F be the set of all functions f such that:

(i) $f: \kappa \times (\omega_1 \times \kappa) \to \mathbf{B}$;

(ii) $\gamma \neq \gamma' \to f(\gamma, (\alpha,\beta)) \wedge f(\gamma', (\alpha,\beta)) = \mathbf{0}$;

(iii) $\gamma \geqslant \beta \to f(\gamma, (\alpha,\beta)) = \mathbf{0}$;

(iv) $|\{z \in \kappa \times (\omega_1 \times \kappa) | f(z) > \mathbf{0}\}| \leqslant \omega_1$;

(v) for some ordinal $\varphi(f) < \omega_1$, $\alpha \geqslant \varphi(f) \to f(\gamma, (\alpha,\beta)) = \mathbf{0}$;

(vi) for all ordinals $\delta < \kappa$, $\mathrm{ran}[f \restriction \delta] \subseteq \mathbf{B}_{\delta^+}$, where $f \restriction \delta$ abbreviates $f \restriction (\delta \times (\omega_1 \times \delta))$ and where δ^+ denotes the first cardinal greater than δ.

Using F, we define a poset P *in* $M[G]$ as follows. For $f \in F$, define $\bar{f}$ (*in* $M[G]$) by

$$\bar{f} = \{(\gamma, (\alpha, \beta)) \mid (\exists p \in G)[p \leqslant_{\mathbf{B}} f(\gamma, (\alpha, \beta))]\}.$$

Let $P = \{\bar{f} \mid f \in F\}$, and partially order P by $f \leqslant_P g \leftrightarrow \bar{f} \supseteq \bar{g}$. Clearly, if $f \in P$, then f is a function such that:

(i) $\mathrm{dom}(f) \subseteq \omega_1 \times \kappa$;

(ii) $(\alpha, \beta) \in \mathrm{dom}(f) \to f(\alpha, \beta) \in \beta$;

(iii) $|f| \leqslant \omega_1$;

(iv) for some ordinal $\psi(f) < \omega_1$, $(\alpha, \beta) \in \mathrm{dom}(f) \to \alpha < \psi(f)$

(v) for all ordinals $\delta < \kappa$, $f \restriction \delta \in M[G_{\delta^+}]$, where $f \restriction \delta$ abbreviates $f \restriction (\omega_1 \times \delta)$.

[Note: P does not, however, contain all such functions. This was pointed out to us by Mitchell in a private communication. However, it is easily seen that P is closed under simple set-theoretical operations such as the union of two compatible members.]

For future use, notice that if $\lambda > \omega_1$ is a regular cardinal in M, then for $f \in P$, $f \upharpoonright \lambda \in M[G_\lambda]$ and for $f \in F$ ran$(f \upharpoonright \lambda] \subseteq \mathbf{B}_\lambda$. (Both of these hold because f is only non-trivial at ω_1 places.)

Recalling Lemma 2.3, we define a poset Q with domain $C \times F$ by setting, *in* M, $(p, f) \leqslant_Q (q, g) \leftrightarrow p \leqslant_c q \;\&\; p \Vdash_C$ "$\overline{f} \leqslant_P \overline{g}$" (i.e., iff $p \supseteq q \;\&\; p \Vdash_C$ "$\overline{f} \supseteq \overline{g}$"). By Lemma 2.3, if K is M-generic for Q with $G = \{p \in C \mid \langle p, 0_F\rangle \in K\}$ (where $0_F = \{\langle 0, z\rangle \mid z \in \kappa \times (\omega_1 \times \kappa)\}$) (which we may assume by Lemma 2.3) and H is defined as $\{\overline{f} \in P \mid \langle \emptyset, f\rangle \in K\}$, then H is $M[G]$-generic for P and $M[K] = M[G][H]$.

Define a partial ordering $\leqslant_F$ on F, *in* M by $f \leqslant_F g \leftrightarrow 1 \Vdash_C$ "$\overline{f} \supseteq \overline{g}$". Clearly, $f \leqslant_F g$ iff for all $z \in \kappa \times (\omega_1 \times \kappa)$, $f(z) \geqslant_{\mathbf{B}} g(z)$.

Suppose that, *in* M, $\delta < \omega_1$ and $\langle f_\alpha \mid \alpha < \delta\rangle$ is a sequence of members of F such that $\alpha < \beta < \delta \rightarrow f_\beta \leqslant_F f_\alpha$. Define $g : \kappa \times (\omega_1 \times \kappa) \rightarrow \mathbf{B}$ by $g(z) = \mathbf{V}^{\mathbf{B}} \{f_\alpha(z) \mid \alpha < \delta\}$ for each $z \in \kappa \times (\omega_1 \times \kappa)$. (Since $\langle f_\alpha \mid \alpha < \delta\rangle \in M$, this supremum in $\mathbf{B}$ always exists.) We write $g = \bigwedge_{\alpha<\delta} f_\alpha$, since it is easily seen that $g \in F$ here, and that $g \leqslant_F f_\alpha$ for all $\alpha < \delta$.

Lemma 4.1. *Let $f, g \in F$ and suppose that $p \Vdash_C$ "$\overline{f} \supseteq \overline{g}$" for some $p \in C$. Then there is $h \in F$ such that $h \leqslant_F g$ and $p \Vdash_C$ "$\overline{h} \supseteq \overline{f}$".*

Proof. For each $z = (\gamma, (\alpha, \beta)) \in \kappa \times (\omega_1 \times \kappa)$, define

$$h(z) = g(z) \vee [f(z) \wedge p \upharpoonright \beta^+].$$

Lemma 4.2. *Suppose $D \in M[G]$ and that D is a dense open subset of P. Then, for any $f \in F$ there is $g \in F$ such that $g \leqslant_F f$ and $\overline{g} \in D$. Moreover, suppose $p \Vdash_C$ "$\mathring{D}$ is a dense open subset of $\mathring{P}$". Then, for any $f \in F$ there is $g \in F$ such that $g \leqslant_F f$ and $p \Vdash_C$ "$\overline{g} \in \mathring{D}$".*

Proof. The first part of the lemma follows both from Lemma 4.1 and from the second part of the lemma. We prove the second part of the lemma by an argument due to Easton.

Working in M, we inductively define a sequence $\langle (p_\alpha, f_\alpha) \mid \alpha < \delta\rangle$, for some $\delta < \omega_1$, such that:

(i) $p_\alpha \in C$, $f_\alpha \in F$, $p_\alpha \leqslant p$, each $\alpha < \delta$;

(ii) $f_\beta \leqslant_F f_\alpha \leqslant_F f$, each $\alpha < \beta < \delta$;

(iii) $p_\alpha \Vdash_C$ "$\overline{f}_\alpha \in \mathring{D}$", each $\alpha < \delta$;

(iv) $p_\alpha \not\perp p_\beta$, each $\alpha < \beta < \delta$.

The ordinal δ will be determined by the termination of the definition, which will occur at some stage before ω_1 (by virtue of condition (iv) and the c.c.c. for C), when $\{p_\alpha \mid \alpha < \delta\}$ is a *maximal* pairwise incompatible subset of $\{q \in C \mid q \leqslant_C p\}$.

Suppose $\langle (p_\beta, f_\beta) \mid \beta < \alpha \rangle$ is defined. Let $q \leqslant_C p$ be incompatible with each p_β, $\beta < \alpha$, and set $h = \bigwedge_{\beta<\alpha} f_\beta$. Since $q \Vdash_C$ "$\mathring{D}$ is a dense subset of $\mathring{P}$" and $\mathbf{1} \Vdash_C$ "$\bar{h} \in \mathring{P}$", we can find $p_\alpha \leqslant_C q$ and $h' \in F$ such that $p_\alpha \Vdash_C$ "$\bar{h}' \in \mathring{D}$" and $p_\alpha \Vdash_C$ "$\bar{h}' \supseteq \bar{h}$". By Lemma 4.1, pick $f_\alpha \in F$ such that $f_\alpha \leqslant_F h$ and $p_\alpha \Vdash_C$ "$\bar{f}_\alpha \supseteq h'$". Since $p_\alpha \Vdash_C$ "$\mathring{D}$ is open in $\mathring{P}$", $p_\alpha \Vdash_C$ "$\bar{f}_\alpha \in \mathring{D}$". Hence (p_α, f_α) is defined as required.

When the definition terminates, set $g = \bigwedge_{\alpha<\delta} f_\alpha$. Thus $g \in F$, $g \leqslant_F f$. We show that $p \Vdash_C$ "$\bar{g} \in \mathring{D}$". It suffices to show that $\{q \in C \mid q \Vdash_C \text{"}\bar{g} \in \mathring{D}\text{"}\}$ is dense below p in C. Let $q \leqslant p$. Thus $q \sim p_\alpha$ for some $\alpha < \delta$. Let $q' \leqslant_C q, p_\alpha$. By condition (iii), $q' \Vdash_C$ "$\bar{f}_\alpha \in \mathring{D}$". So, as $g \leqslant_F f_\alpha$, $q' \Vdash_C$ "$\bar{g} \in \mathring{D}$", and we are done.

Corollary 4.3. *If* $\lambda < \omega_1^M$ *and* $s : \lambda \to M$, $s \in M[K]$, *then* $s \in M[G]$. *In particular,* $\mathcal{P}^{M[K]}(\lambda) = \mathcal{P}^{M[G]}(\lambda)$ *and* $\omega_1^{M[K]} = \omega_1^{M[G]}$ $(= \omega_1^M)$.

Proof. Suppose $p \in G$, $(p, f) \Vdash_Q \mathring{s} : \check{\lambda} \to \check{V}$. For each $\alpha < \lambda$, define D_α in $M[G]$ by

$$D_\alpha = \{\bar{f} \in P \mid (\exists x \in M)[\bar{f} \Vdash_P \text{"}\mathring{s}(\check{\alpha}) = \check{x}\text{"}]\}.$$

Clearly, $p \Vdash_C$ "$\mathring{D}_\alpha$ is a dense open subset of $\mathring{P}$", here, so we can use Lemma 4.2 to define, in M, a sequence $\langle f_\alpha \mid \alpha < \lambda \rangle$ from F so that $\alpha < \beta < \lambda \to f_\beta \leqslant_F f_\alpha$ and $p \Vdash$ "$\bar{f}_\alpha \in \mathring{D}_\alpha$". Set $g = \bigwedge_{\alpha<\lambda} f_\alpha$. Clearly, $(p, g) \leqslant_Q (p, f)$ and $(p, g) \Vdash_Q$ "$\mathring{s} \in \check{V}[\mathring{G}]$".

Lemma 4.4. *Assume* $V = M[G]$. *Then* P *satisfies the* κ*-c.c.*

Proof. The argument is a slight modification of the usual one for the Lévy collapsing poset on an inaccessible. Clause (v) in the definition of F was designed partly to make this argument work, even though κ is only *weakly* inaccessible here.

Let X be a set of pairwise incompatible elements of P. We define, inductively, sequences $\langle X_\alpha \mid \alpha < \omega_2 \rangle$, $\langle \nu_\alpha \mid \alpha < \omega_2 \rangle$, such that

(i) $\alpha < \beta < \omega_2 \to X_\alpha \subseteq X_\beta \subseteq X$ and $\alpha < \omega_2 \to |X_\alpha| < \kappa$;

(ii) $\alpha < \beta < \omega_2 \to \nu_\alpha < \nu_\beta < \kappa$;

(iii) $f \in X_\alpha \to \mathrm{dom}(f) \subseteq \omega_1 \times \nu_\alpha$.

Let $f_0 \in X$ be arbitrary. Set $X_0 = \{f_0\}$, and let ν_0 be the least cardinal such that $\mathrm{dom}(f_0) \subseteq \omega_1 \times \nu_0$. Suppose X_α, ν_α are defined. Let X'_α be a maximal subset of X such that

$$[f, g \in X'_\alpha \;\&\; f \restriction \nu_\alpha = g \restriction \nu_\alpha \to f = g.$$

By definition of P, $\{f \restriction \nu_\alpha \mid f \in X'_\alpha\} \subseteq M[G_{\nu_\alpha^+}]$. By Lemma 2.1, κ is inaccessible in $M[G_{\nu_\alpha^+}]$, so $|\nu_\alpha^{\nu_\alpha}|^{M[G_{\nu_\alpha^+}]} < \kappa$. Hence $|X'_\alpha| < \kappa$. Set $X_{\alpha+1} = X_\alpha \cup X'_\alpha$ and let $\nu_{\alpha+1} < \kappa$ be the least cardinal such that $\nu_{\alpha+1} > \nu_\alpha$ and $f \in X_{\alpha+1} \to \mathrm{dom}(f) \subset \omega_1 \times \nu_{\alpha+1}$. If $\lim(\alpha)$, set $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$, $\nu_\alpha = \sup_{\beta<\alpha} \nu_\beta$. Since κ is weakly inaccessible, $|X_\alpha| < \kappa$ and $\nu_\alpha < \kappa$ here also. This completes the definition. Set $Y = \bigcup_{\alpha<\omega_2} X_\alpha$. Thus $|Y| < \kappa$. We finish by showing that $X \subseteq Y$. Let $f \in X$. As $\langle \nu_\alpha \mid \alpha < \omega_2 \rangle$ is strictly increasing and $|f| \leqslant \omega_1$, we can find $\alpha < \omega_2$ such that $f \restriction \nu_\alpha = f \restriction \nu_{\alpha+1}$. By construction of $X_{\alpha+1}$, there is $g \in X_{\alpha+1}$ such that $f \restriction \nu_\alpha = g \restriction \nu_\alpha$. By (iii), $\mathrm{dom}(g) \subseteq \omega_1 \times \nu_{\alpha+1}$. Hence $f \sim g$, which means $f = g \in Y$, as X is pairwise incompatible.

Corollary 4.5. *Assume $V = M$. Then Q satisfies κ-c.c.*

Proof. A standard argument for two-step forcing. See [6], for example.

Lemma 4.6. $\omega_2^{M[K]} = \kappa$.

Proof. By Corollary 4.3 and Lemma 4.4, it suffices to show that if $\omega_1^M < \lambda < \kappa$, then $|\lambda|^{M[K]} = \omega_1^{M[G]}$. By definition of P in $M[G]$, this clearly reduces to showing that if (*in* $M[G]$) $f \in P$ and $\gamma \in \lambda < \kappa$, there is $g \in P$, $g \supseteq f$, such that for some $\alpha \in \omega_1$, $g(\alpha, \lambda) = \gamma$. But look, $\psi(f) < \omega_1$, so if we pick $\alpha > \psi(f)$, then $g = f \cup \{(\gamma, (\alpha, \lambda))\} \in P$ is clearly as required.

In M, for $\gamma < \kappa$, set $F_\gamma = \{f \restriction \gamma \mid f \in F\}$, $F^\gamma = \{f - f \restriction \gamma \mid f \in F\}$, $Q_\gamma = C_\gamma \times F_\gamma$, $Q^\gamma = C^\gamma \times F^\gamma$. Again, for $\gamma < \kappa$, let $K_\gamma = K \cap Q_\gamma$, $K^\gamma = K \cap Q^\gamma$. In $M[G]$, for $\gamma < \kappa$, set $P_\gamma = \{f \restriction \gamma \mid f \in P\}$, $P^\gamma = \{f - f \restriction \gamma \mid f \in P\}$. Note that whenever $\lambda > \omega_1^M$ is regular in M, then $P_\lambda \in M[G_\lambda]$, and P_λ, P^λ are related to F_λ, F^λ in the same way that P is related to F. Partially order Q^γ *in* $M[G_\lambda]$ by

$$(p, f) \leqslant_{Q^\gamma} (q, g) \leftrightarrow p \leqslant_C q \;\&\; (\exists p' \in G_\lambda)[p' \cup p \Vdash_C \text{“}\check{f} \supseteq \check{g}\text{”}].$$

Then:

Lemma 4.7. *Let $\lambda > \omega_1^M$ be a regular cardinal in M. Then K_λ is M-generic for Q_λ, K^λ is $M[K_\lambda]$-generic for Q^λ, and $M[K_\lambda][K^\lambda] = M[K]$.*

Proof. Set $H_\lambda = H \cap P_\lambda$, $H^\lambda = H \cap P^\lambda$. Since $P \cong P_\lambda \times P^\lambda$ in $M[G]$, Lemma 2.2 tells us that H_λ is $M[G]$-generic for P_λ and H^λ is $M[G][H_\lambda]$-generic for P^λ and

$$M[G][H_\lambda][H^\lambda] = M[G][H] = M[K].$$

Again, C^λ, $P_\lambda \in M[G_\lambda]$, so by Lemma 2.2,

$$M[G][H_\lambda] = M[G_\lambda][G^\lambda][H_\lambda] = M[G_\lambda][H_\lambda][G^\lambda],$$

where G^λ is $M[G_\lambda][H_\lambda]$-generic for C^λ. Hence, by Lemma 2.3,

$$M[K_\lambda][K^\lambda] = M[G_\lambda][H_\lambda][G^\lambda][H^\lambda] = M[K], \text{etc.}$$

The next lemma shows that under certain circumstances there is an element which will play the role of $\bigwedge_{\nu<\delta} f_\nu$ for decreasing sequences of members of F which do not lie in M. (In such cases, we will abuse our notation by writing $\bigwedge_{\nu<\delta} f_\nu$ to denote such an element.)

Lemma 4.8. *Let $\gamma \geqslant \omega_1^M$, $\delta < \omega_1^M$, and let $\langle f_\nu \mid \nu < \delta\rangle$ be a sequence of members of F^γ in $M[K_{\gamma^+}]$ such that $\nu < \tau < \delta \to f_\tau \leqslant_F f_\nu$. Then there is a $g \in F^\gamma$ such that $M[K_{\gamma^+}] \models (\forall \nu < \delta)(1 \Vdash_{C^{\gamma^+}}$ "$\check{g} \supseteq \check{f}_\nu$").*

Proof. By Corollary 4.3, $\langle f_\nu \mid \nu < \delta\rangle \in M[G_{\gamma^+}]$. Let $\mathring{f}$ be a term of the (M, C_{γ^+})-forcing language such that $\mathring{f}^{M[G_{\gamma^+}]} = \langle f_\nu \mid \nu < \delta\rangle$. (Thus $\mathring{f}$ will contain constants of the form $\check{x}$ for $x \in M$ and possibly the constant $\mathring{G}$ which represents G_{γ^+} in $M[G_{\gamma^+}]$.) Pick $p \in G_{\gamma^+}$ such that $p \Vdash_{C_{\gamma^+}}$ "$\mathring{f}$ is a $\check{\delta}$-sequence of members of F^γ such that $\nu < \tau < \check{\delta} \to \mathring{f}(\tau) \leqslant_F \mathring{f}(\nu)$". Work in M. Define a function g by setting, for $z = (\iota, (\alpha, \beta)) \in \kappa \times (\omega_1 \times \kappa)$ with $\beta \geqslant \gamma$,

$$g(z) = p \wedge \bigvee^{\mathbf{B}} \{f(z) \wedge \|\check{f} = \mathring{f}(\check{\nu})\|^{\mathbf{B}_{\gamma^+}} \mid f \in F^\gamma \ \&\ \nu < \delta\}.$$

We show that $g \in F^\gamma$.

We must therefore verify that g satisfies clauses (i)–(vi) in the definition of F. Clause (i) holds by definition.

For clause (ii), suppose $\xi \neq \zeta$ and that $g(\xi, (\alpha, \beta)) \wedge g(\zeta, (\alpha, \beta)) > 0$, some α, β. Thus for some $\nu < \tau < \delta$, and some $f, f' \in F^\gamma$,

$$p \wedge \|\check{f} = \mathring{f}(\check{\nu})\| \wedge \|\check{f}' = \mathring{f}(\check{\tau})\| \wedge f(\xi, (\alpha,\beta)) \wedge f'(\zeta, (\alpha,\beta)) > 0.$$

But look, by choice of p, this means

$$\|\check{f}' \leqslant_F \check{f}\| \wedge f(\xi,(\alpha,\beta)) \wedge f'(\zeta,(\alpha,\beta)) > 0.$$

Hence, clearly,

$$f'(\xi,(\alpha,\beta)) \wedge f'(\zeta,(\alpha,\beta)) > 0,$$

contrary to $f' \in F$. Hence clause (ii) holds for g.

For clause (iii), note that if $\iota \geqslant \beta$, then $f(\iota,(\alpha,\beta)) = 0$ for all $f \in F$, so $g(\iota,(\alpha,\beta)) = 0$. Since C_{γ^+} satisfies c.c.c.,

$$|\{f \mid (\exists \nu < \delta)\,(\|\check{f} = \mathring{f}(\check{\nu})\| > 0)\}| < \omega_1,$$

whence clause (iv) clearly holds.

This last fact also implies that clause (v) holds. Finally, note that clause (vi) holds for g, since we are only working "above" γ here, and g is defined from members of F^γ and certain elements of $\mathbf{B}_{\gamma^+}$. Hence $g \in F^\gamma$. Now we place ourselves in $M[K_{\gamma^+}]$. Let $\nu < \delta$. Thus $\|\check{f}_\nu = \mathring{f}(\check{\nu})\| \in G_{\gamma^+}$. Also $p \in G_{\gamma^+}$, of course. Clearly, therefore, $1 \Vdash_{C\gamma^+}$ "$\check{g} \supseteq \check{f}_\nu$", as required.

Lemma 4.9. *Let $\lambda > \omega_1^M$ be a regular cardinal in M, and let γ be a limit ordinal in M, $\mathrm{cf}^M(\gamma) > \omega$. Let $t \in M[K]$, $t : \gamma \to M$, and suppose that for all $\delta < \gamma$, $t \restriction \delta \in M[K_\lambda]$. Then, in fact, $t \in M[K_\lambda]$.*

Proof. Almost identical to the proof of [7, Lemma 3.8].

Using Lemma 4.9, it is now very easy, using the fact that κ is weakly compact in M, to prove the following result:

Theorem 4.10. $M[K] \models$ *"There are no ω_2-Aronszajn trees".*

Proof. Just as [7, Theorem 5.8].

That completes the first part of the proof. Now we turn to the problem of adapting Silver's argument to the present situation, in order to establish that Δ holds in $M[K]$.

From now on, we shall assume that $M \models \mathrm{MA} + 2^\omega = \omega_2$. By Lemmas 2.5 and 2.1, this causes no loss of generality.

We require a result essentially due to Łoś and Sierpinski. They proved, long ago, that if $\mathfrak{A}$ is any infinite structure with a countable language, then one could find a single binary function f on the domain of $\mathfrak{A}$ such that all of the functions, relations, and constants of $\mathfrak{A}$ could be defined in terms of f. For a proof of this, the reader should see [3, Theorem 3.3]. For our part, this gives us the following useful formulation of Δ.

Lemma 4.11. ZFC $\vdash \Delta$ *iff whenever* $f: \omega_2 \times \omega_2 \to \omega_2$, *there is* $X \subseteq \omega_2$, $|X| = \omega_1$, *such that* $f''X^2 \subseteq X$ *and* $|X \cap \omega_1| = \omega$.

Using Lemma 4.11, we shall show that Δ holds in $M[K]$. Let $t \in M[K]$, $t: \kappa \times \kappa \to \kappa$. Pick $(p_0, f_0) \in G \times F$ so that $(p_0, f_0) \Vdash_Q$ "$\mathring{t}: \check{\kappa} \times \check{\kappa} \to \check{\kappa}$". In $M[G]$, for each $\alpha, \beta < \kappa$, let

$$D_{\alpha\beta} = \{\bar{f} \in P \mid \bar{f} \leqslant_P \bar{f}_0 \ \&\ (\exists \gamma \in \kappa)[\bar{f} \Vdash_P \text{“}\mathring{t}(\check{\alpha},\check{\beta}) = \check{\gamma}\text{”}]\}.$$

Clearly, each $D_{\alpha\beta}$ is a dense open subset of P below $\bar{f}_0$. We may assume that $p_0 \Vdash_C$ "$\langle \mathring{D}_{\alpha\beta} \mid \alpha, \beta \in \check{\kappa}\rangle$ is a sequence of open subsets of $\mathring{P}$ and for each $\alpha, \beta \in \check{\kappa}$, $\mathring{D}_{\alpha\beta}$ is dense below $\bar{f}_0$". In M, for each $\alpha, \beta < \kappa$, let

$$E_{\alpha\beta} = \{f \in F \mid f \leqslant_F f_0 \ \&\ (p_0, f) \Vdash_Q \text{“}\mathring{t}(\check{\alpha},\check{\beta}) = \check{\gamma}\text{” for some } \gamma < \kappa\}.$$

Clearly, $E_{\alpha\beta} = \{f \in F \mid f \leqslant_F f_0 \ \&\ p_0 \Vdash_C \text{“}\check{f} \in \mathring{D}_{\check{\alpha}\check{\beta}}\text{”}\}$, so by Lemma 4.2, $E_{\alpha\beta}$ is a dense open subset of the poset F^*, which has domain $\{f \in F \mid f \leqslant_F f_0\}$ and ordering $\leqslant_F$. Let R be the relation defined by

$$R(f, \alpha, \beta, \gamma) \longleftrightarrow f \in F^* \ \&\ (p_0, f) \Vdash_Q \text{“}\mathring{t}(\check{\alpha},\check{\beta}) = \check{\gamma}\text{”}.$$

Thus $R \in M$. *Work in M from now on.*

Lemma 4.12. F^* *satisfies the* κ-*c.c.*

Proof. By an argument as in Lemma 4.4.

Consider the first-order structure

$$\mathfrak{A} = \langle V_{\kappa^+}, \in, \kappa, \omega_1, F, F^*, \leqslant_F, \psi, R, \{p_0\}, \{f_0\}\rangle,$$

where $\psi: F \to \omega_1$ is the function involved in the definition of F. Let $\mathfrak{A}^*$ be a skolem expansion of $\mathfrak{A}$.

As κ is Ramsey, there is $X \subseteq \kappa$, $|X| = \kappa$ X homogeneous for $\mathfrak{A}^*$. Let Y consist of the first ω_1 members of X. Let W be the universe of the substructure of $\mathfrak{A}^*$ generated by Y. Thus W is the universe of a unique $\mathfrak{B} \prec \mathfrak{A}$.

Let $U = W \cap \kappa$. Since the language of $\mathfrak{A}^*$ is countable, $|U| = \omega_1$.

Lemma 4.13. *The poset* $F^* \restriction W = \langle F^* \cap W, \leqslant_F \cap W^2 \rangle$ *satisfies c.c.c.*

Proof. Suppose not, and let J be a collection of ω_1 pairwise incompatible elements of $F^* \restriction W$. Since the language of $\mathfrak{A}^*$ is countable, we can assume that for some fixed (skolem) term τ,

$$J = \{\tau^{\mathfrak{A}^*}(x_1^\alpha, \dots, x_n^\alpha) \mid x_1^\alpha, \dots, x_n^\alpha \in Y \,\&\, x_1^\alpha < \dots < x_n^\alpha \,\&\, \alpha < \omega_1\}.$$

By a well known combinatorial argument (see [6], for instance), we can assume that for some integer m, $1 \leqslant m < n$, $x_1^\alpha = x_1, \dots, x_m^\alpha = x_m$, where $x_1, \dots, x_m$ are independent of α here, and for all $\alpha < \beta < \omega_1$, $x_n^\alpha < x_{m+1}^\beta$. Pick elements $x_{m+1}^\alpha, \dots, x_n^\alpha$ of X for $\omega_1 \leqslant \alpha < \kappa$ now so that $\alpha < \beta < \kappa \to x_n^\alpha < x_{m+1}^\beta$, with $x_{m+1}^\alpha < \dots < x_n^\alpha$ for each α. Since J is pairwise incompatible in F^*, a simple indiscernibility argument shows that

$$J' = \{\tau^{\mathfrak{A}^*}(x_1, \dots, x_m, x_{m+1}^\alpha, \dots, x_n^\alpha) \mid \alpha < \kappa\}$$

is a set of κ incompatible elements of F^*, contrary to Lemma 4.12.

Lemma 4.14. $|U \cap \omega_1| = \omega$.

Proof. Suppose not. As above, we can find a (skolem) term τ such that

$$U \cap \omega_1 \supseteq \{\tau^{\mathfrak{A}^*}(x_1, \dots, x_m, x_{m+1}^\alpha, \dots, x_n^\alpha) \mid (x_1 < \dots < x_m < x_{m+1}^0)$$

$$\& \ (\alpha < \beta < \omega_1 \to x_{m+1}^\alpha < \dots < x_n^\alpha < x_{m+1}^\beta) \ \& \ (\alpha < \omega_1)$$

$$\& \ (x_1, \dots, x_m \in Y) \ \& \ (\alpha < \omega_1 \to x_{m+1}^\alpha, \dots, x_n^\alpha \in Y)\},$$

where for each $\alpha < \beta < \omega_1$,

$$\tau^{\mathfrak{A}^*}(x_1, \dots, x_m, x_{m+1}^\alpha, \dots, x_n^\alpha) \neq \tau^{\mathfrak{A}^*}(x_1, \dots, x_m, x_{m+1}^\beta, \dots, x_n^\beta).$$

Pick elements $x_{m+1}^\alpha, \dots, x_n^\alpha$ from X for $\omega_1 \leqslant \alpha < \kappa$ as before. For each $\alpha < \omega_1$,

$$\mathfrak{A}^* \models \tau(x_1, \dots, x_m, x_{m+1}^\alpha, \dots, x_n^\alpha) < \omega_1,$$

so by indiscernibility,

$$\{\tau^{\mathfrak{A}^*}(x_1, \dots, x_m, x_{m+1}^\alpha, \dots, x_n^\alpha) \mid \alpha < \kappa\}$$

is a set of κ distinct $\in$-predecessors of ω_1, which is absurd.

Now, for each $\alpha, \beta \in \kappa$, the fact that $E_{\alpha\beta}$ is a dense subset of F^* may be expressed in $\mathfrak{A}$ by the sentence

$$(\forall f \in F^*)(\exists g \in F^*)(\exists \gamma \in \kappa)[g \leqslant_F f \;\&\; R(g, \alpha, \beta, \gamma)].$$

So, as $\mathfrak{B} \prec \mathfrak{A}$, for each $\alpha, \beta \in U$, we have

$$(\forall f \in F^* \restriction W)(\exists g \in F^* \restriction W)(\exists \gamma \in U)[g \leqslant_F f \;\&\; R(g, \alpha, \beta, \gamma)].$$

Thus, if

$$E'_{\alpha\beta} = \{f \in F^* \restriction W \mid (p_0, f) \Vdash_Q \text{“}\mathring{t}(\check{\alpha}, \check{\beta}) = \check{\gamma}\text{” for some } \gamma \in U\}$$

for each $\alpha, \beta \in U$, then $E'_{\alpha\beta}$ is a dense open subset of $F^* \restriction W$. Let $\mathfrak{J} = \{E'_{\alpha\beta} \mid \alpha, \beta \in U\}$. Since $|\mathfrak{J}| = |U| = \omega_1$, by Lemma 4.13 and MA, we can thus find an $\mathfrak{J}$-generic subset S of $F^* \restriction W$. Since S is compatible in F^*, we can define $h : \kappa \times (\omega_1 \times \kappa) \to \mathbf{B}$ by $h(z) = \mathbf{V}^{\mathbf{B}}\{f(z) \mid f \in S\}$. Since $|S| \leqslant \omega_1$, $|\{z \mid h(z) > 0\}| \leqslant \omega_1$. It is easily seen that h satisfies clauses (i), (ii), (iii), (iv), (vi) in the definition of F. Moreover, h satisfies clause (v) also. For, $\mathfrak{A} \models \psi : F \to \omega_1$, so as $\mathfrak{B} \prec \mathfrak{A}$, $\psi : F^* \restriction W \to U \cap \omega_1$. And by Lemma 4.14, $|U \cap \omega_1| = \omega$. Hence, if $\rho = \sup(U \cap \omega_1)$, then $\rho < \omega_1$ and for all $f \in F^* \restriction W$, if $\alpha > \rho$, then $f(\gamma, (\alpha, \beta)) = 0$. So, $\alpha > \rho \to h(\gamma, (\alpha, \beta)) = 0$, as required. Thus $h \in F$. And clearly $h \leqslant_F f$ for all $f \in S$. (In particular, $h \in F^*$.) Thus, as $S \cap E'_{\alpha\beta} \neq \emptyset$ for all $\alpha, \beta \in U$, we see that if $\alpha, \beta \in U$, then there is $\gamma \in U$ such that $(p_0, h) \Vdash_Q$ “$\mathring{t}(\check{\alpha}, \check{\beta}) = \check{\gamma}$”. Hence $(p_0, h) \Vdash_Q$ “$\mathring{t}''\check{U}^2 \subseteq \check{U}$”. We have therefore shown that if $p_0 \Vdash_C$ “$[\check{f}_0 \Vdash_P$ “$\mathring{t} : \check{\kappa} \times \check{\kappa} \to \check{\kappa}$”]”, then there is $h \leqslant_F f_0$ and $U \subseteq \kappa$, $|U| = \omega_1$, $|U \cap \omega_1| = \omega$, such that $p_0 \Vdash_C$ “$[\check{h} \Vdash_P$ “$\mathring{t}''\check{U}^2 \subseteq \check{U}$”]”.

Hence, as $p_0 \in G$ and $f_0 \in F$ was arbitrary such, we have proved the following:

Theorem 4.15. $M[K] \models \Delta$.

Corollary 4.16. $M[K] \models 2^\omega = \omega_2 + \Phi$.

Postscript

The model $M[K]$ constructed above has the following model-theoretic property. In $M[K]$, there is a countable first-order theory T with a two-cardinal model of *type* (ω_1, ω) but no model of type (ω_2, ω_1), and yet any model of type (ω_2, ω_1) (with a countable language) has an *elementary substructure* of type (ω_1, ω). These two properties are, in a sense, precisely counter-intuitive from a model-theoretic point of view; i.e., one usually regards it as "almost true" that every countable T with an (ω_1, ω) model has an (ω_2, ω_1) model and as "almost false" that Δ holds. (The first of these two is, of course, provable under the assumption either that ω_2 is accessible in $L[A]$ for some $A \subseteq \omega_1$, or else that $2^\omega = \omega_1$.)

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